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2002 J. Phys. A: Math. Gen. 35 6329

(http://iopscience.iop.org/0305-4470/35/30/310)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **35** (2002) 6329–6337 PII: S0305-4470(02)36030-X

Dirac equation for a spin-$\frac{1}{2}$ charged particle on the 2D sphere S^2 and the hyperbolic plane H^2

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Received 18 April 2002
Published 19 July 2002
Online at stacks.iop.org/JPhysA/35/6329

Abstract
Using the idea of shape invariance with respect to the main quantum number n, we represent Lie algebras $u(2)$ and $u(1,1)$. The induced metric by the Casimir operator of Lie algebras $u(2)$ and $u(1,1)$ leads us to obtain new solutions of the Dirac equation corresponding to a spin-$\frac{1}{2}$ charged particle on the 2D sphere S^2 and the hyperbolic plane H^2 in the presence of a magnetic monopole. It is shown that the related new spinors represent the supersymmetry algebra, and that they satisfy shape invariance equations with respect to n.

PACS numbers: 02.30.Gp, 03.65.−w, 02.20.Sv, 02.10.Nj

1. Introduction

Shape invariance is one of the most interesting aspects of the theory of exactly solvable systems [1–13] in the framework of supersymmetric quantum mechanics [14–20]. In fact, all solvable problems of quantum mechanics are either supersymmetric or can be made so. In the formalism of shape invariance, partner Hamiltonians are related by supersymmetry transformations. The spectra of two partner Hamiltonians are identical, except for the ground state. Two partner Hamiltonians factorize out into the product of the raising and the lowering operators, and the energy eigenfunctions of these systems transform into each other with the help of these operators [6, 8, 20]. Therefore, shape invariance symmetry provides the possibility of exactly determining the corresponding wavefunction of the problem, using algebraic procedure. However, it has been shown that a wide range of shape invariance potentials lie in *two different classes* [21, 22]. In the *first class*, which is obtained from factorization of the Schrödinger equation with respect to *the main quantum number n*, the superpotential is explained in terms of the master function, the corresponding weight function, *the main quantum number n* and also *the secondary quantum number m* [22] (n and m are non-negative integers and the maximum value of m is equal to n). The *second class* is derived from factorizing the Schrödinger equation with respect to *the secondary quantum number m*, and the superpotential is explained in terms of the master function, its weight function and also *the secondary quantum number m* [21, 22].

When a Schrödinger equation is exactly solvable for a potential, then there always exists a Dirac equation corresponding to it which is exactly solvable. Sukumar has shown how to exploit the supersymmetry along with shape invariance to obtain the complete energy spectrum and eigenfunctions of the Dirac equation corresponding to a 3D Coulomb field [23]. Using the supersymmetry methods developed in the category of 1D shape invariant potentials, in the framework of chiral and complex supersymmetry, the spectrum and the eigenfunctions of the Dirac equation of 2D and 4D Euclidean spaces in the presence of external fields have been studied [24]. Also, using the idea of shape invariance, the relativistic bound-state spectra and spinor wavefunctions of Dirac–Coulomb, Dirac oscillator, Dirac–Morse, Dirac–Rosen–Morse, Dirac–Eckart, Dirac–Pöschl–Teller and Dirac–Scarf potentials have been obtained [25–27]. In [28], using the generators of Lie algebra $gl(2, c)$ obtained from the idea of shape invariance with respect to the parameter m we obtained a solution of the massless Dirac equation for a charged particle with spin-$\frac{1}{2}$ on the homogeneous manifold $SL(2, c)/GL(1, c)$ in the presence of a magnetic monopole field.

In this paper, using the idea of shape invariance with respect to the main quantum number n, based on the master function theory, we introduce new representations for the Lie algebras $u(2)$ and $u(1, 1)$. Then, comparing the Casimir operator of these Lie algebras with the general form of the Laplace–Beltrami operator for a 2D manifold, we obtain the metric of the simplest homogeneous manifolds non-zero constant curvature $S^2 = SU(2)/U(1)$ and $H^2 = SU(1, 1)/U(1)$ in terms of the master function. It is shown that shape invariance with respect to n leads us to solve the massless Dirac equation in Minkowskian spacetime with the space part of S^2 or H^2, in the presence of a magnetic monopole. In fact, we give a new analysis of the integrable systems corresponding to a massless spin-$\frac{1}{2}$ charged particle on the S^2 and H^2. Also, we prove that the spinors of the Dirac equation represent supersymmetry algebra and shape invariance symmetry with respect to n.

2. Towards the new bases of representation for Lie algebras $u(2)$ and $u(1, 1)$

In the master function theory discussed in [21, 29, 30] we choose the master function $A(x)$ as an exactly second order ($A''(x) \neq 0$), and without double root. Also, suppose the non-negative weight function $W(x)$ in an interval (a, b) is $W(x) = A^{\lambda}(x)$. The parameter λ and the interval (a, b) can be determined so that the expression $A^{\lambda+1}(x)$ and all of its derivatives vanish at the end points of the interval. It is clear that the expression $A^{-\lambda}(x)(A^{\lambda+1}(x))' = (\lambda + 1)A'(x)$ is linear in terms of x. Therefore, the required conditions of the master function theory are established. Now using the shape invariance with respect to the parameter n, introduced in [29], we obtain the following factorized differential equations

$$\begin{aligned} B_+(n)B_-(n)\Phi_{n,m}(x) &= E(n, m)\Phi_{n,m}(x) \\ B_-(n)B_+(n)\Phi_{n-1,m}(x) &= E(n, m)\Phi_{n-1,m}(x) \end{aligned} \tag{1}$$

where the explicit forms of the raising and the lowering operators corresponding to the parameter n are, respectively;

$$B_+(n) = A(x)\frac{\mathrm{d}}{\mathrm{d}x} + \frac{1}{2}(n + 2\lambda) A'(x) \qquad B_-(n) = -A(x)\frac{\mathrm{d}}{\mathrm{d}x} + \frac{1}{2}nA'(x). \tag{2}$$

The eigenvalue $E(n,m)$ and the associated special functions are expressed in terms of master function $A(x)$ and the parameter λ as

$$E(n,m) = -(n-m)(n+m+2\lambda)\delta^2 \tag{3}$$

$$\Phi_{n,m}(x) = \frac{a_{n,m}}{A^{\lambda+\frac{m}{2}}(x)}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{n-m} A^{n+\lambda}(x). \tag{4}$$

In the above equations, the constant δ is either a real number or a pure imaginary number, and is defined in terms of the master function parameters as

$$\delta := \sqrt{-\frac{A'^2(0) - 2A''A(0)}{4}} \tag{5}$$

and $a_{n,m}$ is the normalization constant. Now, by choosing the following recursion relation between the normalization coefficients $a_{n,m}$

$$a_{n,m} = \frac{n+m+2\lambda}{2n+2\lambda}\frac{a_{n-1,m}}{\sqrt{E(n,m)}} \tag{6}$$

one may write down the shape invariance equations (1) as the raising and the lowering equations:

$$B_+(n)\Phi_{n-1,m}(x) = \sqrt{E(n,m)}\Phi_{n,m}(x) \tag{7a}$$

$$B_-(n)\Phi_{n,m}(x) = \sqrt{E(n,m)}\Phi_{n-1,m}(x). \tag{7b}$$

To obtain relation (6) we just need to multiply both sides of (7a) by the factor $A^{m/2}(x)$ and then compare the coefficients of x^{n+m} on both sides of the resulting equation.

We shall now use these results to obtain the new differential representations of Lie algebras $u(2)$ and $u(1,1)$ in terms of the master function. Defining the new operators J_+ and J_- as

$$\begin{aligned} J_+ &= \mathrm{e}^{\mathrm{i}\phi}\left[\frac{\partial}{\partial\theta} - \frac{\mathrm{i}A'(x)}{2}\frac{\partial}{\partial\phi} + \frac{1+2\lambda}{2}A'(x)\right]_{x=x(\theta)} \\ J_- &= \mathrm{e}^{-\mathrm{i}\phi}\left[-\frac{\partial}{\partial\theta} - \frac{\mathrm{i}A'(x)}{2}\frac{\partial}{\partial\phi}\right]_{x=x(\theta)} \end{aligned} \tag{8}$$

and the new functions $\Phi_{n,m}(\theta,\phi)$ as

$$\Phi_{n,m}(\theta,\phi) = [\mathrm{e}^{\mathrm{i}n\phi}\Phi_{n,m}(x)]_{x=x(\theta)} \tag{9}$$

and also using the shape invariance equations (1), one can conclude the following factorized equations:

$$\begin{aligned} J_+J_-\Phi_{n,m}(\theta,\phi) &= E(n,m)\Phi_{n,m}(\theta,\phi) \\ J_-J_+\Phi_{n-1,m}(\theta,\phi) &= E(n,m)\Phi_{n-1,m}(\theta,\phi). \end{aligned} \tag{10}$$

The change of variable $x = x(\theta)$ is obtained by solving the first-order differential equation

$$\mathrm{d}\theta = \frac{\mathrm{d}x}{A(x)}. \tag{11}$$

Note that the variable ϕ exists in the interval $0 \leqslant \phi < 2\pi$. Now, it is easy to verify that the operators J_+ and J_- together with the operators

$$J_3 = -\mathrm{i}\frac{\partial}{\partial\phi} \qquad \text{and} \qquad I = 1 \tag{12}$$

constitute the commutative relations of the Lie algebras $u(2)$ and $u(1,1)$ as

$$[J_+, J_-] = 2\delta^2 J_3 + (2\lambda + 1)\delta^2 \qquad [J_3, J_\pm] = \pm J_\pm \qquad [J_\pm, I] = [J_3, I] = 0. \tag{13}$$

For instance, the value of the constant δ is equal to 1 and i for $A(x) = 1 + x^2$ and $A(x) = 1 - x^2$, respectively. Therefore, (13) constitute commutation relations corresponding to the Lie algebras $u(2)$ and $u(1,1)$ for the master functions $A(x) = 1 + x^2$ and $A(x) = 1 - x^2$, respectively. These are the reverse of the results of [30] for the master functions $A(x) = 1+x^2$ and $A(x) = 1 - x^2$. That is commutation relations (2.15) of [30], for $A(x) = 1 + x^2$ and $A(x) = 1 - x^2$, reduce to the commutation relations of Lie algebras $u(1,1)$ and $u(2)$, respectively. From (7), one can conclude that the operators J_+ and J_- satisfy the following raising and lowering relations:

$$\begin{aligned} J_+\Phi_{n-1,m}(\theta,\phi) &= \sqrt{E(n,m)}\Phi_{n,m}(\theta,\phi) \\ J_-\Phi_{n,m}(\theta,\phi) &= \sqrt{E(n,m)}\Phi_{n-1,m}(\theta,\phi). \end{aligned} \tag{14}$$

Results (14) together with

$$J_3\Phi_{n,m}(\theta,\phi) = n\Phi_{n,m}(\theta,\phi) \qquad I\Phi_{n,m}(\theta,\phi) = \Phi_{n,m}(\theta,\phi) \tag{15}$$

state that, for a given m, the new functions $\Phi_{n,m}(\theta,\phi)$ with different values of $n \geqslant m$ are the representation bases of the Lie algebras $u(2)$ and $u(1,1)$. This is the reverse of the result of [30] for *quantum numbers n and m*. The Casimir operator of the generators J_+, J_-, J_3 and I is

$$H = -J_+J_- - \delta^2 J_3^2 - 2\lambda\delta^2 J_3 + \tfrac{1}{2}(2\lambda+1)\delta^2 \tag{16}$$

with the following eigenvalue equation

$$H\Phi_{n,m}(\theta,\phi) = -\left(m^2 + 2m\lambda - \lambda - \tfrac{1}{2}\right)\delta^2\Phi_{n,m}(\theta,\phi). \tag{17}$$

Since the eigenvalue of the Casimir operator is independent of n, and functions $\Phi_{n,m}(\theta,\phi)$ with $n \geqslant m$ are eigenfunctions of Casimir operator H, then we have an infinite-fold degeneracy.

3. New exact solutions for the Dirac equation corresponding to a massless spin-$\frac{1}{2}$ charged particle on the 2D sphere S^2 and the hyperbolic plane H^2

To construct a Schrödinger Hamiltonian, we can determine the metric g_{ij}, the gauge potential A_i and the electic potential V by writing the Casimir operator H as follows in terms of the Laplace–Beltrami operator

$$-\tfrac{1}{2}H = -\tfrac{1}{2}D_j^A D^{Aj} + V. \tag{18}$$

The covariant derivative D_j^A is written in terms of gauge and Levi-Civita connections

$$D_j^A := \nabla_j - \mathrm{i}A_j \tag{19}$$

where the index j takes the values θ and ϕ. Comparing the second-order partial derivatives of both sides of (18) we obtain a 2D manifold with the following metric:

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{2}{A''A(x)} \end{pmatrix}. \tag{20}$$

The non-vanishing components of the Christoffel symbols as well as the Ricci tensor of metric (20) are

$$\Gamma^\theta_{\phi\phi} = \frac{A'(x)}{A''A(x)} \qquad \Gamma^\phi_{\theta\phi} = -\tfrac{1}{2}A'(x) \tag{21a}$$

$$R_{\theta\theta} = \delta^2 \qquad R_{\phi\phi} = \frac{2\delta^2}{A''A(x)}. \tag{21b}$$

The manifold described by metric (20) has the constant Ricci scalar curvature:

$$R = 2\delta^2. \tag{22}$$

Similar to [30], we can easily show that metric (20) can be derived from an appropriate parametrization in terms of the variables θ and ϕ for the homogeneous manifolds $S^2 = SU(2)/U(1)$ and $H^2 = SU(1,1)/U(1)$. From Ricci scalar curvature (22), it is clear that the described manifolds by the metric (20) are S^2 and H^2 for the master functions $A(x) = 1 + x^2$ and $A(x) = 1 - x^2$, respectively.

Let us now consider the $1+2$ Minkowskian spacetime metric as

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -\frac{2}{A''A(x)} \end{pmatrix}. \tag{23}$$

Indices μ and ν denote rows and columns by the time variable t, space variables θ and ϕ. It is evident, of course, that just for the $1+2$ Minkowskian spacetime metric (23), the non-vanishing components of the Christoffel symbols and the Ricci tensor are calculated similar to (21).

The generators of the Clifford algebra that generate $1+2$ diagonal metric $\eta^{ab} := (1, -1, -1)$ by the following equation [31]

$$\gamma^a\gamma^b = \eta^{ab} I_{2\times 2} - \mathrm{i}\epsilon^{abc}\gamma_c \tag{24}$$

are defined as the matrices

$$\gamma^0 = \sigma^3 \qquad \gamma^1 = \mathrm{i}\sigma^2 \qquad \gamma^2 = -\mathrm{i}\sigma^1 \tag{25}$$

where σ^1, σ^2 and σ^3 are the known Pauli matrices. It is obvious that indices a, b and c take the values 0, 1 and 2. The massless Dirac operator of $1+2$ Minkowskian spacetime corresponding to the metric (23) and Clifford generators (25) is defined as

$$D_{1+2} = -\mathrm{i}\gamma^a E_a{}^\mu \left(\partial_\mu - \mathrm{i}A_\mu + \tfrac{1}{8}\omega_{\mu\, bc}[\gamma^b, \gamma^c]\right). \tag{26}$$

A_t is a scalar potential corresponding to an electric field, A_θ and A_ϕ are the components of a gauge potential corresponding to a magnetic field. The 3-bein $E_a{}^\mu$ and its inverse, i.e. $e_\mu{}^a$, establish a connection between the Minkowskian diagonal metric η_{ab} and the spacetime metric $g_{\mu\nu}$ (as well as their inverse):

$$E_a{}^\mu \eta^{ab} E_b{}^\nu = g^{\mu\nu} \qquad E_a{}^\mu g_{\mu\nu} E_b{}^\nu = \eta_{ab} \tag{27a}$$

$$e_\mu{}^a g^{\mu\nu} e_\nu{}^b = \eta^{ab} \qquad e_\mu{}^a \eta_{ab} e_\nu{}^b = g_{\mu\nu}. \tag{27b}$$

Using (27) the 3-beins for the metric (23) are calculated as

$$E_a{}^\mu = \left(e_\mu{}^a\right)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \sqrt{\frac{A''A(x)}{2}} \end{pmatrix}. \tag{28}$$

Also, using (21a) and (28) together with the equation

$$\partial_\mu e_\nu{}^a - \Gamma^\lambda_{\mu\nu} e_\lambda{}^a + \omega_\mu{}^a{}_b e_\nu{}^b = 0 \tag{29}$$

we calculate the non-vanishing components of the spin connection

$$\omega_{\phi\ 12} = -\omega_{\phi\ 21} = \frac{A'(x)}{\sqrt{2A''A(x)}}. \tag{30}$$

Now the massless time-dependent Dirac equation corresponding to a spin-$\frac{1}{2}$ charged particle on the 2D sphere S^2 and the hyperbolic plane H^2 in the presence of a magnetic field and an electric field can be written as follows

$$D_{1+2}\Psi(t;\theta,\phi)=0 \tag{31}$$

or

$$-\mathrm{i}\begin{pmatrix} \frac{\partial}{\partial t}-\mathrm{i}A_t & -\frac{\partial}{\partial\theta}-\mathrm{i}\sqrt{\frac{A''A(x)}{2}}\frac{\partial}{\partial\phi}+\mathrm{i}A_\theta \\ & -\sqrt{\frac{A''A(x)}{2}}A_\phi+\frac{1}{4}A'(x) \\ \frac{\partial}{\partial\theta}-\mathrm{i}\sqrt{\frac{A''A(x)}{2}}\frac{\partial}{\partial\phi}-\mathrm{i}A_\theta & -\frac{\partial}{\partial t}+\mathrm{i}A_t \\ -\sqrt{\frac{A''A(x)}{2}}A_\phi-\frac{1}{4}A'(x) & \end{pmatrix}\Psi(t;\theta,\phi)=0. \tag{32}$$

To obtain (32) we have substituted (25), (28) and (30) in (26). In this paper we assume that the electric field does not exist, i.e. $A_t=0$, and that the magnetic field is static, i.e. $\frac{\partial A_\mu}{\partial t}=0$, with the axial symmetry $A_\mu=A_\mu(\theta)$ for $\mu=\theta$ and ϕ. Therefore, if we assume the time evolution and the spatial components of the spinors as

$$\Psi(t;\theta,\phi)=\mathrm{e}^{-\mathrm{i}\sqrt{E(n,m)}t}\begin{pmatrix}\psi_1(\theta,\phi)\\ \mathrm{i}\psi_2(\theta,\phi)\end{pmatrix} \tag{33}$$

then from the Dirac equation (32) we get the following two equations:

$$\begin{gathered}\left(\frac{\partial}{\partial\theta}-\mathrm{i}\sqrt{\frac{A''A(x)}{2}}\frac{\partial}{\partial\phi}-\mathrm{i}A_\theta-\sqrt{\frac{A''A(x)}{2}}A_\phi-\frac{1}{4}A'(x)\right)\psi_1(\theta,\phi)=\sqrt{E(n,m)}\psi_2(\theta,\phi)\\ \left(-\frac{\partial}{\partial\theta}-\mathrm{i}\sqrt{\frac{A''A(x)}{2}}\frac{\partial}{\partial\phi}+\mathrm{i}A_\theta-\sqrt{\frac{A''A(x)}{2}}A_\phi+\frac{1}{4}A'(x)\right)\psi_2(\theta,\phi)=\sqrt{E(n,m)}\psi_1(\theta,\phi).\end{gathered} \tag{34}$$

Let us assume that the functionality of ϕ is some phase factor for the functions $\psi_1(\theta,\phi)$ and $\psi_2(\theta,\phi)$. It is easy to show that the existence of difference in phases between the functions $\psi_1(\theta,\phi)$ and $\psi_2(\theta,\phi)$ has no effect on the determination of the magnetic field. Therefore, we choose the same phase factor as follows

$$\psi_1(\theta,\phi)=\mathrm{e}^{\mathrm{i}k\phi}\psi_1(\theta)\qquad \psi_2(\theta,\phi)=\mathrm{e}^{\mathrm{i}k\phi}\psi_2(\theta). \tag{35}$$

To compare the derived results with (7), we reduce equations (34) with respect to ϕ. Thus, by considering relation (11) we obtain

$$\begin{aligned}B_+(n)&=\left[\frac{\mathrm{d}}{\mathrm{d}\theta}-\mathrm{i}A_\theta(\lambda;\theta)+(k-A_\phi(n,\lambda;\theta))\sqrt{\frac{A''A(x)}{2}}-\frac{1}{4}A'(x)\right]_{x=x(\theta)}\\ B_-(n)&=\left[-\frac{\mathrm{d}}{\mathrm{d}\theta}+\mathrm{i}A_\theta(\lambda;\theta)+(k-A_\phi(n,\lambda;\theta))\sqrt{\frac{A''A(x)}{2}}+\frac{1}{4}A'(x)\right]_{x=x(\theta)}\end{aligned} \tag{36}$$

with

$$A_\theta(\lambda;\theta)=\left[\frac{\mathrm{i}}{4}(1+2\lambda)A'(x)\right]_{x=x(\theta)}\qquad A_\phi(\lambda,n;\theta)=\left[k-\frac{(n+\lambda)A'(x)}{\sqrt{2A''A(x)}}\right]_{x=x(\theta)}. \tag{37}$$

The 2-form of the magnetic field corresponding to the gauge potential (37) is calculated as

$$B(\lambda,n;\theta)=\left[\frac{-2(n+\lambda)\delta^2}{\sqrt{2A''A(x)}}\right]_{x=x(\theta)}\mathrm{d}\theta\wedge\mathrm{d}\phi. \tag{38}$$

It can be proved that the equation

$$\frac{\delta}{\tan\delta\theta} = -\frac{A'(x)}{2} \tag{39}$$

is the solution of the differential equation (11). It is evident that using the change of variable (39), one may obtain the Minkowskian spacetime metric (23) as

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -\frac{\sin^2\delta\theta}{\delta^2} \end{pmatrix} \tag{40}$$

where it is related to the sphere S^2 and the hyperplane H^2 depending upon which one of the values $\delta = 1$ and $\delta = \mathrm{i}$ is chosen, respectively. Also, the 2-form of the magnetic field (38) becomes

$$B(\lambda, n; \theta) = -(n+\lambda)\delta^2 \frac{\sin\delta\theta}{\delta}\, \mathrm{d}\theta \wedge \mathrm{d}\phi. \tag{41}$$

Thus, in this paper, the magnetic monopole field (41) has been quantized in terms of *the main quantum number* $n \geqslant m$. This is the reverse of the result of [28], because the magnetic monopole field had been quantized there in terms of *the secondary quantum number* $m \leqslant n$. Also, the above comparison clearly indicates that the spinors are labelled by n and m as

$$\begin{aligned} \Psi_{n,m}(t; \theta, \phi) &= \mathrm{e}^{-\mathrm{i}\sqrt{E(n,m)}t} \begin{pmatrix} \psi_{n-1,m}(\theta, \phi) \\ \mathrm{i}\psi_{n,m}(\theta, \phi) \end{pmatrix} \\ &=: \mathrm{e}^{-\mathrm{i}\sqrt{E(n,m)}t}\, \Psi_{n,m}(\theta, \phi) \end{aligned} \tag{42}$$

where we have used the following notation:

$$\psi_{n,m}(\theta, \phi) = \mathrm{e}^{\mathrm{i}k\phi}\, \Phi_{n,m}(x(\theta)). \tag{43}$$

It is clear that apart from a phase factor $\mathrm{e}^{\mathrm{i}k\phi}$, the components of the spinors on the S^2 and H^2 are expressed in terms of the associated special functions, the hyperbolic Gegenbauer and Gegenbauer:

$$\Phi_{n,m}(x) \longrightarrow \mathcal{P}^{(\lambda)}_{n,m}(x) = \frac{a_{n,m}}{(1+x^2)^{\lambda+\frac{m}{2}}} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{n-m} (1+x^2)^{n+\lambda} \tag{44a}$$

$$\Phi_{n,m}(x) \longrightarrow P^{(\lambda)}_{n,m}(x) = \frac{a_{n,m}}{(1-x^2)^{\lambda+\frac{m}{2}}} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{n-m} (1-x^2)^{n+\lambda} \tag{44b}$$

respectively. The spinors $\Psi_{n,m}(t; \theta, \phi)$ describe the motion of a spin-$\frac{1}{2}$ charged particle in the presence of the magnetic monopole field (41) on the 2D sphere S^2 and the hyperbolic plane H^2, depending upon which one of the master functions $A(x) = 1 + x^2$ and $A(x) = 1 - x^2$ is chosen. The matrix components of the time-independent Dirac equation (31) can be written as

$$\begin{aligned} b_+(\lambda, n)\psi_{n-1,m}(\theta, \phi) &= \sqrt{E(n, m)}\psi_{n,m}(\theta, \phi) \\ b_-(\lambda, n)\psi_{n,m}(\theta, \phi) &= \sqrt{E(n, m)}\psi_{n-1,m}(\theta, \phi) \end{aligned} \tag{45}$$

where the raising and lowering operators of the spinor components are

$$\begin{aligned} b_+(\lambda, n) &= \frac{\partial}{\partial\theta} - \mathrm{i}\frac{\delta}{\sin\delta\theta}\frac{\partial}{\partial\phi} - \mathrm{i}A_\theta(\lambda; \theta) - \frac{\delta}{\sin\delta\theta}A_\phi(\lambda, n; \theta) + \frac{\delta}{2\tan\delta\theta} \\ b_-(\lambda, n) &= -\frac{\partial}{\partial\theta} - \mathrm{i}\frac{\delta}{\sin\delta\theta}\frac{\partial}{\partial\phi} + \mathrm{i}A_\theta(\lambda; \theta) - \frac{\delta}{\sin\delta\theta}A_\phi(\lambda, n; \theta) - \frac{\delta}{2\tan\delta\theta}. \end{aligned} \tag{46}$$

Equations (45) represent 2D shape invariance equation components of spinors with respect to *the main quantum number* n on the S^2 and H^2 for $A(x) = 1 + x^2$ ($\delta = 1$) and $A(x) = 1 - x^2$ ($\delta = \mathrm{i}$), respectively.

4. Representation of N = 1 chiral supersymmetry algebra and shape invariance symmetry by the Dirac new spinors

Now, we show that the spinors $\Psi_{n,m}(\theta, \phi)$ represent a supersymmetry algebra $N = 1$, and also a shape invariance symmetry with respect to *the main quantum number n*. Using (45), we can easily see that they satisfy the Dirac eigenvalue equation

$$D_2(\lambda, n)\Psi_{n,m}(\theta, \phi) = \sqrt{E(n, m)}\Psi_{n,m}(\theta, \phi) \tag{47}$$

in which the time-independent Dirac operator $D_2(\lambda, n)$ is

$$D_2(\lambda, n) := \begin{pmatrix} 0 & -\mathrm{i}b_-(\lambda, n) \\ \mathrm{i}b_+(\lambda, n) & 0 \end{pmatrix}. \tag{48}$$

The square of the Dirac operator

$$D_2^2(\lambda, n) = \begin{pmatrix} b_-(\lambda, n)b_+(\lambda, n) & 0 \\ 0 & b_+(\lambda, n)b_-(\lambda, n) \end{pmatrix} =: H(\lambda, n) \tag{49}$$

leads to two partner Hamiltonian on S^2 and H^2 with shape invariance symmetry on n. For the fermionic creation and annihilation operators [24]

$$Q_\pm(\lambda, n) := Q_\pm b_\mp(\lambda, n) \tag{50}$$

with

$$Q_\pm := \mp\frac{\mathrm{i}}{2}(\sigma^1 \pm \mathrm{i}\sigma^2) \tag{51}$$

one can obtain the following chiral decomposition for the time-independent Dirac operator:

$$D_2(\lambda, n) = Q_+(\lambda, n) + Q_-(\lambda, n). \tag{52}$$

Therefore, it becomes obvious that two of the fermionic operators $Q_+(\lambda, n)$ and $Q_-(\lambda, n)$ together with the bosonic operator $H(\lambda, n)$, i.e. the square of the time-independent Dirac operator, satisfy the supersymmetry algebra $N = 1$ as

$$Q_\pm^2(\lambda, n) = 0 \qquad H(\lambda, n) = \{Q_+(\lambda, n), Q_-(\lambda, n)\} \qquad [Q_\pm(\lambda, n), H(\lambda, n)] = 0. \tag{53}$$

The representation of supersymmetry algebra on S^2 and H^2 by the spinors $\Psi_{n,m}(\theta, \phi)$ is

$$\begin{aligned} Q_+(\lambda, n)\Psi_{n,m}(\theta, \phi) &= \sqrt{E(n, m)}\begin{pmatrix} \psi_{n-1,m}(\theta, \phi) \\ 0 \end{pmatrix} \\ Q_-(\lambda, n)\Psi_{n,m}(\theta, \phi) &= \sqrt{E(n, m)}\begin{pmatrix} 0 \\ \mathrm{i}\psi_{n,m}(\theta, \phi) \end{pmatrix} \\ H(\lambda, n)\Psi_{n,m}(\theta, \phi) &= E(n, m)\Psi_{n,m}(\theta, \phi). \end{aligned} \tag{54}$$

The appropriate operators for representing the shape invariance symmetry on S^2 and H^2 by the spinors $\Psi_{n,m}(\theta, \phi)$ are

$$\mathcal{B}_\pm(\lambda, n) := \begin{pmatrix} \sqrt{\frac{E(n,m)}{E(n-1,m)}}b_\pm(\lambda, n-1) & 0 \\ 0 & b_\pm(\lambda, n) \end{pmatrix}. \tag{55}$$

With the help of (45) one can easily conclude the shape invariance equations on 2D manifolds S^2 and H^2 for the spinors $\Psi_{n,m}(\theta, \phi)$:

$$\begin{aligned} \mathcal{B}_+(\lambda, n)\Psi_{n-1,m}(\theta, \phi) &= \sqrt{E(n, m)}\Psi_{n,m}(\theta, \phi) \\ \mathcal{B}_-(\lambda, n)\Psi_{n,m}(\theta, \phi) &= \sqrt{E(n, m)}\Psi_{n-1,m}(\theta, \phi) \end{aligned} \tag{56}$$

or

$$\begin{aligned} \mathcal{B}_+(\lambda, n)\mathcal{B}_-(\lambda, n)\Psi_{n,m}(\theta, \phi) &= E(n, m)\Psi_{n,m}(\theta, \phi) \\ \mathcal{B}_-(\lambda, n)\mathcal{B}_+(\lambda, n)\Psi_{n-1,m}(\theta, \phi) &= E(n, m)\Psi_{n-1,m}(\theta, \phi). \end{aligned} \tag{57}$$

5. Conclusion

Therefore, using the idea of shape invariance with respect to *the main quantum number n*, we derived new bases of representation for Lie algebras $u(2)$ and $u(1,1)$. Also, with the help of the theory of a master function we have obtained the new solutions (42) for the Dirac equation corresponding to a massless spin-$\frac{1}{2}$ charged particle on the 2D sphere S^2 and the hyperbolic plane H^2 in the presence of the magnetic monopole field (41) quantized by *the main quantum number n*. Time-independent spinors corresponding to these solutions represent supersymmetry algebra $N = 1$ as (54). They also realize the representation of shape invariance symmetry with respect to *the main quantum number n* as (56) and (57).

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